Large deviation principles for first-order scalar conservation laws with stochastic forcing

Dong Zhao
Academy of Mathematics and Systems Sciences, CAS
with: J.L. Wu, R.R. Zhang, T.S. Zhang

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Outline

1. Deterministic scalar conservation law
   - Definition of all kinds of solutions
2. Stochastic scalar conservation law
   - Definition of all kinds of solutions
3. LDP for stochastic scalar conservation law
   - Freidlin-Wentzell's large deviations and statement of the main result
   - Skeleton equations
   - Large deviations

Outline

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   - Definition of all kinds of solutions
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Deterministic scalar conservation law

The scalar conservation law is written as

$$\begin{cases} du + div(A(u))dt = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(0) = u_0 & \text{on } \mathbb{R}. \end{cases} \tag{1}$$

- 'scalar' is necessary.
- Different (boundary and initial value) conditions, different type of solutions!

  (1) classical solution

  (2) weak solution

  (3) entropy solution (renormalized entropy solution)

  (4) kinetic solution (generalized kinetic solution)

The Cauchy problem: classical solutions

The homogeneous scalar conservation law

$$\begin{cases} du(x,t) + div(A(u(x,t)))dt = 0, \ x \in \mathbb{R}^m, \ t > 0. \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^m. \end{cases}$$

The flux $A(u) = (A_1(u), \cdots, A_m(u))$ is a given smooth function on $\mathbb{R}$.

Theorem 1 (Dafermos (2005)[7])

Assume u_0 defined on $\mathbb{R}^m$, is bounded and Lipschitz continuous. Let

$$\kappa = ess \inf_{y \in \mathbb{R}^m} divA'(u_0(y)).$$

Then there exists a classical solution u of (1) on the maximal interval $[0, T_\infty)$, where $T_\infty = \infty$ if $\kappa \geq 0$ and $T_\infty = -\kappa^{-1-}$ if $\kappa < 0$. Furthermore, if u_0 is C^k so is u.

- $u(-x,-t)$ is solution $\Leftrightarrow$ $u(x,t)$ is solution.

The Cauchy problem: weak solutions

Definition 1 (Weak solution)

A locally bounded, measurable function u defined on $\mathbb{R}^m \times [0, T)$ is called weak solution of (1), if it satisfies that

$$\int_0^T \int_{\mathbb{R}^m} [\partial_t \phi u + \sum_{i=1}^m \partial_i \phi A_i(u)] dx dt + \int_{\mathbb{R}^m} \phi(x, 0) u_0(x) dx = 0.$$

for every Lipschitz test function ϕ with compact support in $\mathbb{R}^m \times [0, T)$.

- Classical solution is weak solution
- Weak solutions are not unique!

Example: Consider the Cauchy problem for the Burgers equation ($A(u) = \frac{1}{2}u^2$) with initial data

$$u(x,0) = \begin{cases} -1-, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

The above problem admits infinity many weak solutions, including the family

$$u_\alpha(x,t) = \begin{cases} -1-, & -\infty < x \leq t \\ \frac{x}{t}, & -t < x \leq -\alpha t \\ -\alpha, & -\alpha t < x \leq 0 \\ \alpha, & 0 < x \leq t \\ \frac{x}{t}, & \alpha t < x \leq t \\ 1, & t < x < \infty \end{cases}$$

for any $\alpha \in [0,1]$.

How to solve the problem of uniqueness?

To solve the non-uniqueness, additional restrictions in the form of admissibility conditions shall be imposed on weak solutions, which should satisfy the following requirements:

- They should be motivated by physics.
- They should be compatible with other established admissibility conditions.
- They should be broad enough to allow for existence of admissible solutions and sufficiently narrow to single out a unique admissible solution.

Entropy-entropy flux pair—entropy condition

The entropy-entropy flux pair (η, Q): η is an entropy (convex function), with associated entropy flux

$$Q(u) = \int_0^u \eta'(\omega)A'(\omega)d\omega.$$

The entropy condition can be written as:

$$\partial_t \eta(u(x,t)) + divQ(u(x,t)) \leq 0 \tag{2}$$

in the sense of distributions, on $\mathbb{R}^m \times [0, T)$, for every convex entropy-entropy flux pair (η, Q).

- Exclude $u(-x, -t)$.
- For any weak solution $u(x, t)$ satisfying the entropy admissible criterion, the left hand side of (2) is a non-positive distribution, and thereby a measure.
- Try to know as much information on $\eta(u)$ as possible, which implies the property of u.

Admissible weak solution

With the help of the entropy condition (2), the following definition is introduced.

Definition 2 (Admissible weak solution)

A bounded measurable function u on $\mathbb{R}^m \times [0, \infty)$ is an admissible weak solution of (1) with u_0 in $L^\infty(\mathbb{R}^m)$, if the inequality

$$\int_0^\infty \int_{\mathbb{R}^m} [\partial_t \phi \eta(u) + \sum_{i=1}^m \partial_i \phi Q_i(u)] dxdt + \int_{\mathbb{R}^m} \phi(x, 0) \eta(u_0(x)) dx \geq 0$$

holds for every convex entropy-entropy flux pair (η, Q), and all nonnegative Lipschitz continuous test functions ϕ on $\mathbb{R}^m \times [0, \infty)$, with compact support.

Remarks:

- all classical solutions are admissible.
- $\eta(u) = \pm u$, $Q(u) = \pm A(u)$ shows that any admissible weak solution is in particular a weak solution.

Weak entropy solution

A special entropy-entropy flux pairs: Kruzkov's type (1973).

$$
\begin{aligned}
\eta(u;\bar{u}) &= (u-\bar{u})^+, \ \ Q(u;\bar{u}) = sgn(u-\bar{u})^+(A(u)-A(\bar{u})). \\
\eta(u;\bar{u}) &= |u-\bar{u}|, \ \ Q(u;\bar{u}) = sgn(u-\bar{u})(A(u)-A(\bar{u})).
\end{aligned}
$$

Applying the Kruzkov's type entropy-entropy flux pairs, the following definition was introduced by Carrillo(1999)[4].

Definition 3 (Weak entropy solution, Carrillo (1999)[4])

An entropy solution of (2) is a function $u \in L^1(\mathbb{R}^m)$ with $A(u) \in L^1(\mathbb{R}^m)^m$ satisfying

$$
\int_{\{u>k\}} [(u-k)\partial_t \phi + \sum_{i=1}^{m} \partial_i \phi (A_i(u) - A_i(k))] dxdt + \int_{\mathbb{R}^m} (u_0 - k)^+ \phi(0,x) dx \geq 0,
$$

$$
\int_{\{k>u\}} [(k-u)\partial_t \phi + \sum_{i=1}^{m} \partial_i \phi (A_i(k) - A_i(u))] dxdt + \int_{\mathbb{R}^m} (k - u_0)^+ \phi(0,x) dx \geq 0,
$$

for any $(k,\phi) \in \mathbb{R} \times \mathcal{D}((0,T) \times \mathbb{R}^m)$, $\phi \geq 0$.

Renormalized entropy solution

When the regularity of the initial value is relaxed from L^∞ to L^1, the following definition is introduced.

Definition 4 (Renormalized entropy solution)

Let $u_0 \in L^1(\mathbb{R}^m)$, a renormalized entropy solution of (1) is a function $u \in L^1(\mathbb{R}^m \times (0,T))$ such that, for all $k, l \in \mathbb{R}$, the functionals $\mu_{k,l}$ and $\nu_{k,l}$ are Radon measures on $[0,T) \times \mathbb{R}^m$ satisfying

$$\lim_{l \to +\infty} \mu_{k,l}^+([0,T) \times \mathbb{R}^m) = 0 \textit{ and } \lim_{l \to -\infty} \nu_{k,l}^+([0,T) \times \mathbb{R}^m) = 0 \ \forall k \in \mathbb{R}.$$

$$\phi \in \mathcal{D}(\mathbb{R}^m \times (-\infty, T)) \mapsto \mu_{k,l}(\phi) = -\int_{(0,T)\times\mathbb{R}^m} sign_0^+(u \wedge l - k)\{(u \wedge l - k)\phi_t$$

$$+(A(u \wedge l) - A(k)) \cdot \nabla\phi\} dxdt - \int_{\mathbb{R}^m} (u_0 \wedge l - k)^+ \phi(0,x) dx,$$

$$\phi \in \mathcal{D}(\mathbb{R}^m \times (-\infty, T)) \mapsto \nu_{k,l}(\phi) = -\int_{(0,T)\times\mathbb{R}^m} sign_0^+(k - u \wedge l)\{(k - u \wedge l)\phi_t$$

$$+(A(k) - A(u \wedge l)) \cdot \nabla\phi\} dxdt - \int_{\mathbb{R}^m} (k - u_0 \wedge l)^+ \phi(0,x) dx.$$

Definition 5—Kinetic solution

Kinetic formulation of weak entropy solution of a general multidimensional scalar conservation law with Cauchy problem is derived by Lions, Perthame and Tadmor in (1999)[19]. They also proved the equivalence between entropy solutions and the kinetic system.
For the equations

$$\partial_t u + divA(u) = 0 \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^m),$$

the family of entropy inequalities

$$\partial_t \eta(u) + divQ(u) \le 0 \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^m), \tag{3}$$

with Lipschitz continuous convex entropy-entropy flux pair (η, Q) is equivalent to the following kinetic formulation

$$\partial_t \chi(k;u) + \sum_{\alpha=1}^{m} G'_\alpha(k)\partial_\alpha \chi(k;u) = \partial_k m(t,x,k) \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^m\times\mathbb{R}), \tag{4}$$

where

$$\chi(k;u) = \begin{cases} +1, & \text{for } 0 < k \le u, \\ -1, & \text{for } u \le k < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Outline

SSCL–bounded domain and torus

Let $T > 0$ and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,T]}, (\beta_k(t)))$ be a stochastic basis with expectation $\mathbb{E}$.
Model (I) Let D be a bounded open set in $\mathbb{R}^N$ with boundary ∂D in which we assume the boundary ∂D is Lipschitz in case the space dimension $N > 1$. For any $T > 0$, set $Q = (0, T) \times D$ and $\Sigma = (0, T) \times \partial D$. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ be a given probability space and consider

$$\begin{cases} du + divA(u)dt = h(u)dw(t), & \text{in } \Omega \times Q, \\ u(0, \cdot) = u_0(\cdot), & \text{in } D, \\ u = a, & \text{on } \Sigma. \end{cases} \tag{5}$$

a is a random scalar-valued function.
Model (II) Let $T > 0$ and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,T]}, (\beta_k(t)))$ be a stochastic basis with expectation $\mathbb{E}$. We consider the first-order scalar conservation law with stochastic forcing

$$\begin{cases} du + div(A(u))dt = \Phi(u)dW(t), & \text{in } \mathbb{T}^N \times (0, T], \\ u(0) = u_0, & \text{on } \mathbb{T}^N \times \{0\}. \end{cases} \tag{6}$$

The equation is periodic in the space variable $x \in \mathbb{T}^N$, where $\mathbb{T}^N$ is the $N-$dimension torus.

Known results

- Kim (Indiana Univ. Math. J. (2003) [18]) studied the Cauchy problem for the stochastic equation driven by additive noise, wherein the author proposed a method of compensated compactness to prove the existence of a stochastic weak entropy solution via vanishing viscosity approximation. Moreover, a Kruzkov-type method was used to prove the uniqueness.
- Vallet and Wittbold (Infin. Dimens. Anal. Quantum Probab. Relat. Top (2009)[27]) extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. By utilising the vanishing viscosity method, Young measure techniques and Kruzkov doubling variables technique, they managed to show the existence and uniqueness of the stochastic entropy solutions.
- Feng and Nualart (J. Funct. Anal.(2008) [16]) concerned the case of multiplicative noise, for Cauchy problem over the whole spatial space, where they introduced a notion of strong entropy solutions to prove the uniqueness for the entropy solution. Using the vanishing viscosity and compensated compactness arguments, they established the existence of stochastic strong entropy solutions only in 1D case.

- Using a kinetic formulation, Debussche and Vovelle (J. Funct. Anal. (2010)[12]) solved the Cauchy problem for (6) in any dimension. They made use of a notion of kinetic solution developed by Lions, Perthame and Tadmor (J. of A.M.S (2010)[19]) for deterministic first-order scalar conservation laws. In view of the equivalence between kinetic formulation and entropy solution, they obtained the existence and uniqueness of entropy solution.
- The long-time behavior of periodic scalar first-order conservation laws with additive stochastic forcing under an hypothesis of non-degeneracy of the flux function is studied by Debussche and Vovelle (Probab. Theory Related Fields(2015) [13]). For sub-cubic fluxes, they show the existence of an invariant measure. Moreover, for sub-quadratic fluxes, they prove the uniqueness and ergodicity of the invariant measure.
- Mariani (Probab. Theory Related Fields(2010) [22]) (see also (2007)[23] for more) is the first work towards large deviations for stochastic conservation laws, wherein the author considered a family of stochastic conservation laws as parabolic SPDEs with additional small viscosity term and small (spatially) regularized (i.e., spatially smoothing) noises.

Stochastic entropy solution for model (I)

Definition 6 (Stochastic entropy solution)

A function $u \in N^2_\omega(0, T; L^2(D))$ is an entropy solution of stochastic conservation law (5) with the initial condition $u_0 \in L^p(D)$ and boundary condition $a \in L^\infty(\Sigma)$, if $u \in L^2(0, T; L^2(\Omega; L^p(D)))$, $p \geq 2$ and

$$\mu_{\eta,k}(\phi) \geq 0, \quad \mu_{\check{\eta},k}(\phi) \geq 0 \quad dP - a.s.,$$

where $\mu_{\eta,k}(\phi)$ and $\mu_{\check{\eta},k}(\phi)$ are defined in (7) and (8) and $\phi \in \mathcal{D}^+([0, T) \times \mathcal{R}^N)$.

For any function u of $N^2_\omega(0, T; L^2(D))$, any real number k and any regular function $\eta \in \mathcal{E}^+$ (nonnegative convex function in $C^{2,1}(\mathbb{R})$), denote $dP - a.s.$ in Ω by $\mu_{\eta,k}$, the distribution in D defined by

$$\begin{aligned} \mu_{\eta,k}(\phi) &= \int_D \eta(u_0 - k)\phi(0)dx + \int_Q \eta(u-k)\partial_t\phi + \eta'(u-k)(A(u) - A(k)) \cdot \nabla\phi dxdt \\ &\quad + \int_Q \eta'(u-k)h(u)\phi dxdw(t) + \frac{1}{2}\int_Q \eta''(u-k)h^2(u)\phi dxdt \\ &\quad + \int_\Sigma \eta'(a-k)\phi\omega^+(x, k, a(t,x))dSdt, \end{aligned} \tag{7}$$

$$\mu_{\breve{\eta},k}(\phi) = \int_D \breve{\eta}(u_0 - k)\phi(0)dx + \int_Q \breve{\eta}(u-k)\partial_t\phi + \breve{\eta}'(u-k)(A(u)-A(k))\cdot\nabla\phi dxdt$$
$$+ \int_Q \breve{\eta}'(u-k)h(u)\phi dxdw(t) + \frac{1}{2}\int_Q \breve{\eta}''(u-k)h^2(u)\phi dxdt$$
$$+ \int_\Sigma \breve{\eta}'(a-k)\phi\omega^-(x,k,a(t,x))dSdt. \quad (8)$$

Under the following three assumptions:

- (H1) The flux $A : \mathbb{R} \to \mathbb{R}^N$ is of C^2, its derivatives have at most polynomial growth, $A(0) = 0$;
- (H2) $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with $h(0) = 0$;
- (H3) $u_0 \in L^p(D)$, $p \geq 2$ and $a \in L^\infty(\Sigma)$.

Theorem 3 (Lv, Duan, Gao, Wu, (2016) [20])

Under assumptions (H1)-(H3), there exists a unique stochastic entropy solution in the sense of Definition 6.

Renormalized stochastic entropy solution for model **(I)**

Definition 7 (Renormalized stochastic entropy solution)

Let $a \in \mathcal{M}(\Sigma)$ with $\bar{A}(a,x) \in L^1(\Sigma)$ and $u_0 \in L^1(D)$. A function u of $L^1(\Omega; L^1(Q))$ is said to be a renormalized stochastic entropy solution of conservation law of (5), if $\mu_{k,l}(\cdot)$ and $\nu_{k,l}(\cdot)$ defined by (9) and (10) are random measure on $[0,T] \times \bar{D}$ satisfying

$$\lim_{l \to +\infty} \mathbb{E}\mu^+_{k,l}([0,T] \times \bar{D}) = 0, \quad \textit{and} \quad \lim_{l \to -\infty} \mathbb{E}\nu^+_{k,l}([0,T] \times \bar{D}) = 0, \forall k \in \mathbb{R}.$$

For a continuous flux function $A : \mathbb{R} \mapsto \mathbb{R}^N$ and for any measurable boundary data $a : \Sigma \mapsto \mathbb{R}$ with $\bar{A}(a,x) \in L^1(\Sigma)$ where $\bar{A} : \mathbb{R} \times \partial D \mapsto \mathbb{R}$ is defined by

$$\bar{A}(s,x) := \sup\{|A(r) \cdot \vec{n}(x)|, r \in [-s^-, s^+]\}$$

For all $k, l \in \mathbb{R}$, for any $\xi \in \mathcal{D}^+([0,T) \times \mathbb{R}^N)$, define the functionals

$$\mu_{k,l}(\xi) = -\int_D \eta(u_0 \wedge l - k)^+ \xi(0) dx - \int_Q (u \wedge l - k)^+ \xi_t dxdt$$
$$- \int_Q sgn_0^+(u \wedge l - k)(A(u \wedge l) - A(k)) \cdot \nabla \xi dxdt$$
$$- \int_Q sgn_0^+(u \wedge l - k) h(u \wedge l) \xi dxdw(t) - \frac{1}{2} \int_Q [1 - sgn_0^+(k - u \wedge l)] h^2(k) \xi dxdt$$
$$- \int_\Sigma sgn_0^+(a \wedge l - k) \xi \omega^+(x, k, a \wedge l) dSdt \quad dP - a.s., \tag{9}$$

$$\nu_{k,l}(\xi) = -\int_D \eta(k - u_0 \wedge l)^+ \xi(0) dx - \int_Q (k - u \vee l)^+ \xi_t dxdt$$
$$- \int_Q sgn_0^+(k - u \vee l)(A(k) - A(u \vee l)) \cdot \nabla \xi dxdt$$
$$- \int_Q sgn_0^+(k - u \vee l) h(u \vee l) \xi dxdw(t) - \frac{1}{2} \int_Q [1 - sgn_0^+(u \vee l - k)] h^2(k) \xi dxdt$$
$$- \int_\Sigma sgn_0^+(k - a \vee l) \xi \omega^-(x, k, a \vee l) dSdt \quad dP - a.s., \tag{10}$$

Theorem 4 (Lv and Wu (2016)[21])

Let $a \in \mathcal{M}(\Sigma)$ with $\bar{A}(a, x) \in L^1(\Sigma)$ and $u_0 \in L^1(D)$. Under assumptions (H1)-(H2), there exists a unique renormalized stochastic entropy solution in the sense of Definition 7.

For the relationship between stochastic entropy solution and renormalized stochastic entropy solution, we have

Proposition 1 (Lv and Wu (2016)[21])

If u is a stochastic entropy solution in the sense of Definition 6, then u is a renormalized stochastic entropy solution in Definition 7.

Kinetic solution for model **(II)**

Definition 8 (Kinetic solution) Let $u_0 \in L^\infty(\mathbb{T}^N)$. A measurable function $u : \mathbb{T}^N \times [0, T] \to \mathbb{R}$ is said to be a kinetic solution to (6) with initial datum u_0, if $(u(t))$ is predictable, if

$$\mathbb{E}(ess \sup_{t\in[0,T]} \|u(t)\|_{L^1(\mathbb{T}^N)}) \leq C$$

and if there exists a kinetic measure m such that $f(x,t,\xi) := I_{u(x,t)>\xi}$ satisfies: for all $\varphi \in C_c^1(\mathbb{T}^N \times [0,T] \times \mathbb{R})$,

$$\begin{aligned}
&\int_0^T \langle f(t), \partial_t\varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f(t), a(\xi)\cdot\nabla\varphi(t)\rangle dt \\
= \quad & -\sum_{k\geq 1}\int_0^T \int_{\mathbb{T}^N} g_k(x)\varphi(x,t,u(x,t))dxd\beta_k(t) \\
& -\frac{1}{2}\sum_{k\geq 1}\int_0^T \int_{\mathbb{T}^N} \partial_\xi\varphi(x,t,u(x,t))G^2(x)dxdt + m(\partial_\xi\varphi), \ a.s.,
\end{aligned}$$

where $f_0(x,\xi) = I_{u_0(x)>\xi}$.

Kinetic measure

(Kinetic measure) We say that a map m from Ω to the set of non-negative finite measures over $\mathbb{T}^N \times [0, T] \to \mathbb{R}$ is a kinetic measure if

1. m is measurable, in the sense that for each $\phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, $\langle m, \phi \rangle : \Omega \to \mathbb{R}$ is measurable,
2. m vanishes for large ξ: if $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$, then
$$\lim_{R \to +\infty} \mathbb{E} m(\mathbb{T}^N \times [0, T] \times B_R^c) = 0,$$
3. for all $\phi \in C_b(\mathbb{T}^N \times \mathbb{R})$, the process
$$t \mapsto \int_{\mathbb{T}^N \times [0,t] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi)$$
is predictable.

Generalized kinetic solution for model **(II)**

Definition 9 (Generalized kinetic solution) Let $f_0 : \mathbb{T}^N \times \mathbb{R} \to [0,1]$ be a kinetic function. A measurable function $f : \mathbb{T}^N \times [0,T] \times \mathbb{R} \to [0,1]$ is said to be a generalized solution to (6) with the initial datum f_0, if $(f(t))$ is predictable and is a kinetic function such that

$$\mathbb{E}(ess \sup_{t\in[0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi| d\nu_{x,t}(\xi) dx) \leq C,$$

where $\nu := -\partial_\xi f$ and if there exists a kinetic measure m such that for all $\varphi \in C_c^1(\mathbb{T}^N \times [0,T] \times \mathbb{R})$,

$$\begin{aligned} &\int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ = \quad & -\sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x) \varphi(x,t,\xi) d\nu_{x,t}(\xi) dx d\beta_k(t) \\ & -\frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x,t,\xi) G^2(x) d\nu_{x,t}(\xi) dx dt + m(\partial_\xi \varphi), \;\; a.s. \end{aligned}$$

Young measure and Kinetic function

(Young measure) Let (X, λ) be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$. We say that a map $z \mapsto \nu_z(\phi)$ from X to $\mathbb{R}$ is measurable. We say that a Young measure ν vanishes at infinity if, for every $p \geq 1$,

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) < \infty.$$

(Kinetic function) Let (X, λ) be a finite measure space. A measurable function $f : X \times \mathbb{R} \to [0, 1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for $\lambda - a.e.\ z \in X$, for all $\xi \in \mathbb{R}$,

$$f(z, \xi) = \nu_z(\xi, +\infty).$$

We say that f is an equilibrium if there exists a measurable function $u : X \to \mathbb{R}$ such that $f(z, \xi) = I_{u(z) > \xi}$ a.e., or equivalently, $\nu_z = \delta_{u(z)}$ for $a.e.\ z \in X$.

Outline

Thinking of LDP for SSCL

♣

- Which model?
- Which solution?
- Which solution space?
- Which method to use to prove LDP?

♣

- Model **(II)**
- Kinetic solution
- $L^1([0, T]; L^1(\mathbb{T}^N))$
- The weak convergence method.

Thinking of LDP for SSCL

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- $L^1([0,T];L^1(\mathbb{T}^N))$
- The weak convergence method.

Two important characteristics of SSCL:

- No viscosity term and the derivatives of the flux A have at most polynomial growth
- How to calculate the norm in the unusual solution space $L^1([0,T];L^1(\mathbb{T}^N))$?

♠ Adding a viscosity term and using the method of truncation.

♠ Using the doubling variation method.

$$\int_{\mathbb{R}} I_{u^1>\xi}\overline{I_{u^2>\xi}}d\xi = (u^1-u^2)^+, \quad \int_{\mathbb{R}} \overline{I_{u^1>\xi}}I_{u^2>\xi}d\xi = (u^1-u^2)^-.$$

Recall the model (II)

Let $T > 0$ and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,T]}, (\beta_k(t)))$ be a stochastic basis with expectation $\mathbb{E}$. We consider the first-order scalar conservation law with stochastic forcing

$$\begin{cases} du + div(A(u))dt = \Phi(u)dW(t), & \text{in } \mathbb{T}^N \times (0,T], \\ u(0) = u_0, & \text{on } \mathbb{T}^N \times \{0\}. \end{cases} \tag{11}$$

- The flux function A in (11) is supposed to be of class C^2: $A \in C^2(\mathbb{R}; \mathbb{R}^N)$ and its derivatives have at most polynomial growth.
- Assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is complete and W is a cylindrical Wiener process on U: $W = \sum_{k\geq 1} \beta_k e_k$, where β_k are independent Brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in the Hilbert space U.
- The map $\Phi(u) : U \to L^2(\mathbb{T}^N)$ is defined by $\Phi(u)e_k = g_k(u)$ where $g_k(\cdot, u)$ is a regular function on $\mathbb{T}^N$.

Recall the kinetic solution of (11)

Definition 8 (Kinetic solution)

Let $u_0 \in L^\infty(\mathbb{T}^N)$. A measurable function $u : \mathbb{T}^N \times [0,T] \times \Omega \to \mathbb{R}$ is called a kinetic solution to (11) with initial datum u_0, if

1. $(u(t))_{t\in[0,T]}$ is predictable,

2. for any $p \geq 1$, there exists $C_p \geq 0$ such that $\mathbb{E}\Big(ess\sup_{t\in[0,T]} \|u(t)\|^p_{L^p(\mathbb{T}^N)}\Big) \leq C_p$,

3. there exists a kinetic measure m such that $f := I_{u>\xi}$ satisfies the following

$$\begin{aligned}
&\int_0^T \langle f(t), \partial_t\varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f(t), a(\xi)\cdot\nabla\varphi(t)\rangle dt \\
= \; & -\sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N} g_k(x)\varphi(x,t,u(x,t))dxd\beta_k(t) \\
& -\frac{1}{2}\sum_{k\geq 1}\int_0^T\int_{\mathbb{T}^N} \partial_\xi\varphi(x,t,u(x,t))G^2(x)dxdt + m(\partial_\xi\varphi), \;\; a.s.,
\end{aligned}$$

for all $\varphi \in C_c^1(\mathbb{T}^N \times [0,T] \times \mathbb{R})$, where $u(t) = u(\cdot,t,\cdot)$, $G^2 = \sum_{k=1}^\infty |g_k|^2$ and $a(\xi) := A'(\xi)$.

Recall the generalized kinetic solution of (11)

Definition 9 (Generalized kinetic solution)

Let $f_0 : \Omega \times \mathbb{T}^N \times \mathbb{R} \to [0,1]$ be a kinetic function with $(X, \lambda) = (\Omega \times \mathbb{T}^N, P \otimes dx)$. A measurable function $f : \Omega \times \mathbb{T}^N \times [0,T] \times \mathbb{R} \to [0,1]$ is said to be a generalized kinetic solution to (11) with initial datum f_0, if

1. $(f(t))_{t\in[0,T]}$ is predictable,

2. f is a kinetic function with $(X, \lambda) = (\Omega \times \mathbb{T}^N \times [0,T], P \otimes dx \otimes dt)$ and for any $p \geq 1$, there exists a constant $C_p > 0$ such that $\nu := -\partial_\xi f$ fulfills the following $\mathbb{E}\Big(ess\sup_{t\in[0,T]} \int_{\mathbb{T}^N}\int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi)dx\Big) \leq C_p$,

3. There exists a kinetic measure m such that for $\varphi \in C_c^1(\mathbb{T}^N \times [0,T] \times \mathbb{R})$,

$$
\begin{aligned}
&\int_0^T \langle f(t), \partial_t\varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f(t), a(\xi)\cdot\nabla\varphi(t)\rangle dt \\
= \quad & -\sum_{k\geq 1}\int_0^T \int_{\mathbb{T}^N}\int_{\mathbb{R}} g_k(x)\varphi(x,t,\xi)d\nu_{x,t}(\xi)dxd\beta_k(t) \\
& -\frac{1}{2}\int_0^T \int_{\mathbb{T}^N}\int_{\mathbb{R}} \partial_\xi\varphi(x,t,\xi)G^2(x)d\nu_{x,t}(\xi)dxdt + m(\partial_\xi\varphi), \ a.s..
\end{aligned}
$$

Hypothesis

Hypothesis H The flux function A belongs to $C^2(\mathbb{R};\mathbb{R}^N)$ and its derivatives have at most polynomial growth. For each $u \in \mathbb{R}$, the map $\Phi(u) : U \to H$ is defined by $\Phi(u)e_k = g_k(\cdot, u)$, where each $g_k(\cdot, u)$ is a regular function on $\mathbb{T}^N$. More precisely, we assume that $g_k \in C(\mathbb{T}^N \times \mathbb{R})$ with the following bounds

$$\begin{aligned} G^2(x,u) = \sum_{k\geq 1} |g_k(x,u)|^2 &\leq D_0(1+|u|^2), \\ \sum_{k\geq 1} |g_k(x,u) - g_k(y,v)|^2 &\leq D_1\Big(|x-y|^2 + |u-v| J(|u-v|)\Big), \end{aligned}$$

for $x, y \in \mathbb{T}^N$, $u, v \in \mathbb{R}$, and J being a continuous non-decreasing function on $\mathbb{R}^+$ with $J(0) = 0$. Since $\|g_k\|_H \leq \|g_k\|_{C(\mathbb{T}^N)}$, we deduce that $\Phi(u) : U \to H$ is a Hilbert-Schmidt operator, for each $u \in \mathbb{R}$. Hence,

$$\sum_{k\geq 1} \|g_k(\cdot, u)\|_H^2 \leq D_0(1+|u|^2).$$

Global well-posedness of SCL

The following result was shown in [12](2010).

Theorem 5 (Existence, Uniqueness, Reduction)

Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume Hypothesis H holds. Then there is a unique kinetic solution with initial datum u_0 to equation (11). Besides, any generalized kinetic solution f is actually a kinetic solution in the sense that if f is a generalized kinetic solution to (11) with initial datum $I_{u_0>\xi}$, then there exists a kinetic solution u to (11) with initial datum u_0 such that $f(x,t,\xi) = I_{u(x,t)>\xi}$ $a.s.$ for $a.e.$ (x,t,ξ).

Remark 1

The kinetic solution u is a strong solution in the probabilistic sense.

LDP and weak convergence method

Let $\{X^\varepsilon\}_{\varepsilon>0}$ be a family of random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in some Polish space $\mathcal{E}$.

(Rate function) A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if the level set $\{x \in \mathcal{E} : I(x) \le M\}$ is compact for each $M < \infty$.

(Large deviation principle) The sequence $\{X^\varepsilon\}$ is said to satisfy the large deviation principle with rate function I if for each Borel subset A of $\mathcal{E}$

$$-\inf_{x\in A^o} I(x) \le \liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in A) \le \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in A) \le -\inf_{x\in \bar{A}} I(x),$$

where A^o and $\bar{A}$ denote the interior and closure of A in $\mathcal{E}$, respectively.

- An important tool for studying the Freidlin-Wentzell's LDP is the weak convergence approach, which is developed by Dupuis and Ellis in [10].

- The key idea of this approach is to prove certain variational representation formula about the Laplace transform of bounded continuous functionals, which then leads to the verification of the equivalence between the LDP and the Laplace principle. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué and Dupuis in [2] and by Budhiraja and Dupuis in [3].

Suppose $W(t)$ is a cylindrical Wiener process on a Hilbert space U defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ (that is, the paths of W take values in $C([0,T];\mathcal{U})$, where $\mathcal{U}$ is another Hilbert space such that the embedding $U \subset \mathcal{U}$ is Hilbert-Schmidt).

Define

$$\mathcal{A} \triangleq \{\phi : \phi \text{ is a } U-\text{valued } \{\mathcal{F}_t\}\text{predictable process with } \int_0^T |\phi(s)|_U^2 ds < \infty \ \mathbb{P}\text{-}a.s.\}$$

$$S_M \triangleq \{h \in L^2([0,T];U) : \int_0^T |h(s)|_U^2 ds \le M\};$$

$$\mathcal{A}_M \triangleq \{\phi \in \mathcal{A} : \phi(\omega) \in S_M,\ \mathbb{P}\text{-}a.s.\}.$$

We will always refer to the weak topology on the set S_M.

Suppose for each $\varepsilon > 0, \mathcal{G}^\varepsilon : C([0,T];\mathcal{U}) \to \mathcal{E}$ is a measurable map and let $X^\varepsilon := \mathcal{G}^\varepsilon(W)$. Now, we list below sufficient conditions for the large deviation principle of the sequence X^ε as $\varepsilon \to 0$.

Condition A There exists a measurable map $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to \mathcal{E}$ such that the following conditions hold

(a) For every $M < \infty$, let $\{h^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$. If h_ε converges to h as S_M-valued random elements in distribution, then $\mathcal{G}^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h^\varepsilon(s)ds)$ converges in distribution to $\mathcal{G}^0(\int_0^\cdot h(s)ds)$.

(b) For every $M < \infty$, the set $K_M = \{\mathcal{G}^0(\int_0^\cdot h(s)ds) : h \in S_M\}$ is a compact subset of $\mathcal{E}$.

Theorem 6 (Budhiraja et al. (2000)[3])

If $\{\mathcal{G}^\varepsilon\}$ satisfies condition A, then X^ε satisfies the large deviation principle on $\mathcal{E}$ with the following good rate function I defined by

$$I(f) = \inf_{\{h \in L^2([0,T];U) : f = \mathcal{G}^0(\int_0^\cdot h(s)ds)\}} \left\{ \frac{1}{2} \int_0^T |h(s)|_U^2 ds \right\}, \quad \forall f \in \mathcal{E}. \tag{12}$$

By convention, $I(f) = \infty$, if $\left\{ h \in L^2([0,T];U) : f = \mathcal{G}^0(\int_0^\cdot h(s)ds) \right\} = \emptyset$.

A new criterion for LDP

Recently, a sufficient condition to verify **Condition A** is proposed by Matoussi, Sabbagh and Zhang in [24](2008), which turns out to be more suitable for SPDEs arising from fluid mechanics.

Condition B There exists a measurable map $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to \mathcal{E}$ such that the following two items hold

(i) For every $M < +\infty$, and for any family $\{h^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_M$ and any $\delta > 0$,

$$\lim_{\varepsilon \to 0} P(\rho(Y^\varepsilon, Z^\varepsilon) > \delta) = 0,$$

where $Y^\varepsilon := \mathcal{G}^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h^\varepsilon(s)ds)$, $Z^\varepsilon := \mathcal{G}^0(\int_0^\cdot h^\varepsilon(s)ds)$, and $\rho(\cdot,\cdot)$ stands for the metric in the space $\mathcal{E}$.

(ii) For every $M < +\infty$ and any family $\{h^\varepsilon; \varepsilon > 0\} \subset S_M$ that converges to some element h as $\varepsilon \to 0$, $\mathcal{G}^\varepsilon(\int_0^\cdot h^\varepsilon(s)ds)$ converges to $\mathcal{G}^0(\int_0^\cdot h(s)ds)$ in the space $\mathcal{E}$ in Probability.

Compare **Condition A** with **Condition B**

- **Condition B** implies **Condition A**, whose proof can be found in [24].

♠ (ii) in **Condition B** is stronger than (b) in **Condition A**.
However, in previous articles that used Condition A to prove large deviations, we actually obtained Condition B. Hence, there is no differences in verification for stochastic evolution equations.

♠ (i) in **Condition B** is weaker than (a) in **Condition A**.
It reduces the difficulty bought by h^{ε} weak convergence to h when verifying (i) in Condition B. As usual, (i) is not difficult to check because the small noise disappears when $\varepsilon \to 0$.

♢ In fact, we have tried both methods. It turns out this new sufficient condition (**Condition B)** is suitable for establishing LDP for SSCL. In the following, we will point out the difficulty we encountered when verifying Condition A.

Statement of the main result

We consider the following stochastic conservation law driven by small multiplicative noise

$$\begin{cases} du^{\varepsilon} + div(A(u^{\varepsilon}))dt = \sqrt{\varepsilon}\Phi(u^{\varepsilon})dW(t), \\ u^{\varepsilon}(0) = u_0, \end{cases}$$

for $\varepsilon > 0$, where $u_0 \in L^{\infty}(\mathbb{T}^N)$. Under Hypothesis H, by Theorem 5, there exists a unique kinetic solution $u^{\varepsilon} \in L^1([0,T]; L^1(\mathbb{T}^N))$ a.s.. Therefore, there exists a Borel-measurable function

$$\mathcal{G}^{\varepsilon} : C([0,T];\mathcal{U}) \to L^1([0,T]; L^1(\mathbb{T}^N))$$

such that $u^{\varepsilon}(\cdot) = \mathcal{G}^{\varepsilon}(W(\cdot))$.
Let $h \in L^2([0,T]; U)$, we consider the following skeleton equation

$$\begin{cases} du_h + div(A(u_h))dt = \Phi(u_h)h(t)dt, \\ u_h(0) = u_0. \end{cases} \tag{13}$$

The solution u_h, whose existence will be proved in the next section, defines a measurable mapping $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to L^1([0,T]; L^1(\mathbb{T}^N))$ so that $\mathcal{G}^0(\int_0^{\cdot} h(s)ds) := u_h(\cdot)$.

Theorem 7 (DWZZ [9] (2018))

Let $u_0 \in L^{\infty}(\mathbb{T}^N)$. Assume Hypothesis H holds. Then u^{ε} satisfies the large deviation principle on $L^1([0,T]; L^1(\mathbb{T}^N))$ with the good rate function I given by (12).

Global well-posedness of the skeleton equation

Fix $h \in S_M$, and assume $h(t) = \sum_{k\geq 1} h^k(t) e_k$. Now, we introduce definitions of solution to the skeleton equation (13).

Definition 11 (Kinetic solution)

Let $u_0 \in L^\infty(\mathbb{T}^N)$. A measurable function $u_h : \mathbb{T}^N \times [0,T] \to \mathbb{R}$ is said to be a kinetic solution to (13), if for any $p \geq 1$, there exists $C_p \geq 0$ such that

$$ess \sup_{t\in[0,T]} \|u_h(t)\|^p_{L^p(\mathbb{T}^N)} \leq C_p,$$

and if there exists a kinetic measure $m_h \in \mathcal{M}_0^+(\mathbb{T}^N \times [0,T] \times \mathbb{R})$ such that $f_h := I_{u_h > \xi}$ satisfies that for all $\varphi \in C_c^1(\mathbb{T}^N \times [0,T] \times \mathbb{R})$,

$$\int_0^T \langle f_h(t), \partial_t \varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f_h(t), a(\xi)\cdot \nabla\varphi(t)\rangle dt$$
$$= -\sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} g_k(x,\xi)\varphi(x,t,u_h(x,t))h^k(t)dxdt + m_h(\partial_\xi \varphi),$$

where $f_0(x,\xi) = I_{u_0(x)>\xi}$.

Definition 12 (Generalized kinetic solution)

Let $f_0 : \mathbb{T}^N \times \mathbb{R} \to [0,1]$ be a kinetic function. A measurable function $f_h : \mathbb{T}^N \times [0, T] \times \mathbb{R} \to [0,1]$ is said to be a generalized kinetic solution to (13) with the initial datum f_0, if $(f_h(t)) = (f_h(t,\cdot,\cdot))$ is a kinetic function such that for all $p \geq 1$, $\nu^h := -\partial_\xi f_h$ satisfies

$$ess \sup_{t\in[0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu^h_{x,t}(\xi) dx \leq C_p,$$

where C_p is a positive constant and there exists a kinetic measure $m_h \in \mathcal{M}^+_0(\mathbb{T}^N \times [0, T] \times \mathbb{R})$ such that for all $\varphi \in C^1_c(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} &\int_0^T \langle f_h(t), \partial_t \varphi(t)\rangle dt + \langle f_0, \varphi(0)\rangle + \int_0^T \langle f_h(t), a(\xi)\cdot \nabla\varphi(t)\rangle dt \\ &= -\sum_{k\geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x,\xi)\varphi(x,t,\xi) h^k(t) d\nu^h_{x,t}(\xi) dx dt + m_h(\partial_\xi \varphi). \qquad (14) \end{aligned}$$

Theorem 8 (Existence of solutions to skeleton equations, DWZZ [9])

Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume Hypothesis H holds, then for any $T > 0$, (13) has a generalized kinetic solution f_h with initial datum $f_0 = I_{u_0 > \xi}$.

The proof of Theorem 8 is similar to the proof of Theorem 5 which was done in [12], we therefore omit it here.

For the uniqueness, we firstly prove a comparison theorem for two generalized solutions f_i, $i = 1, 2$ of the following equations (15) by using the doubling variation method.

$$\begin{cases} du_h^i + div(A(u_h^i))dt = \Phi(u_h^i)h(t)dt, \\ u_h^i(0) = u_0. \end{cases} \tag{15}$$

Proposition 2 (Comparison Theorem)

Under Hypothesis H, for $0 \le t \le T$, and nonnegative test functions $\rho \in C^\infty(\mathbb{T}^N)$, $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned} & \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta) f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)d\xi d\zeta dxdy \\ \le \quad & \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi-\zeta) f_{1,0}(x,\xi)\bar{f}_{2,0}(y,\zeta)d\xi d\zeta dxdy + K_1 + K_2, \end{aligned}$$

where

$$\begin{aligned} K_1 \quad &:= \quad \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x,s,\xi)\bar{f}_2(y,s,\zeta)(a(\xi)-a(\zeta))\psi(\xi-\zeta)d\xi d\zeta \cdot \nabla_x \rho(x-y)dxdyds \\ K_2 \quad &:= \quad \sum_{k\ge 1} \int_0^t \int_{(\mathbb{T}^N)^2} \rho(x-y) \int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)(g_{k,1}(x,\xi) - g_{k,2}(y,\zeta))h^k(s)d\nu_{x,s}^1 \otimes d\nu_{y,s}^2(\xi,\zeta)dxdyds \end{aligned}$$

with $\gamma_1(\xi,\zeta) = \int_{-\infty}^{\xi} \psi(\xi'-\zeta)d\xi' = \int_{-\infty}^{\xi-\zeta} \psi(y)dy$.

Proof. Let $\varphi_1 \in C_c^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi)$ and $\varphi_2 \in C_c^\infty(\mathbb{T}_y^N \times \mathbb{R}_\zeta)$. Taking a test function of the form $(x, s, \xi) \to \varphi_i(x, \xi)\gamma(s)$ in (14), where γ is the function

$$\gamma(s) = \begin{cases} 1, & s \leq t, \\ 1 - \frac{s-t}{\varepsilon}, & t \leq s \leq t + \varepsilon, \\ 0, & t + \varepsilon \leq s, \end{cases}$$

and letting $\varepsilon \to 0$, we obtain for all $t \in [0, T]$,

$$\begin{aligned} \langle f_1^+(t), \varphi_1\rangle &= \langle f_{1,0}, \varphi_1\rangle + \int_0^t \langle f_1(s), a(\xi) \cdot \nabla_x \varphi_1(s)\rangle ds \\ &+ \sum_{k\geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_{k,1}(x, \xi)\varphi_1(x, \xi)h^k(s)d\nu_{x,s}^1(\xi)dxds - \langle m_1, \partial_\xi \varphi_1\rangle([0, t]), \end{aligned}$$

where $f_{1,0} = I_{u_0 > \xi}$ and $\nu_{x,s}^1(\xi) = -\partial_\xi f_1^+(s, x, \xi)$. Similarly,

$$\begin{aligned} \langle \bar{f}_2^+(t), \varphi_2\rangle &= \langle \bar{f}_{2,0}, \varphi_2\rangle + \int_0^t \langle \bar{f}_2^+(s), a(\zeta) \cdot \nabla_y \varphi_2(s)\rangle ds \\ &- \sum_{k\geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_{k,2}(y, \zeta)\varphi_2(y, \zeta)h^k(s)d\nu_{y,s}^2(\zeta)dyds + \langle m_2, \partial_\zeta \varphi_2\rangle([0, t]). \end{aligned}$$

where $f_{2,0} = I_{u_0 > \zeta}$ and $\nu_{y,s}^2(\zeta) = \partial_\zeta \bar{f}_2^+(s, y, \zeta)$.

Denote the duality distribution over $\mathbb{T}^N_x \times \mathbb{R}_\xi \times \mathbb{T}^N_y \times \mathbb{R}_\zeta$ by $\langle\langle \cdot, \cdot \rangle\rangle$. Setting $\alpha(x,\xi,y,\zeta) = \varphi_1(x,\xi)\varphi_2(y,\zeta)$ and using the integration by parts formula, we have

$$\begin{aligned}
&\langle\langle f_1^+(t)\bar{f}_2^+(t), \alpha\rangle\rangle \\
= \quad & \langle\langle f_{1,0}\bar{f}_{2,0}, \alpha\rangle\rangle + \int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1\bar{f}_2(a(\xi)\cdot\nabla_x + a(\zeta)\cdot\nabla_y)\alpha d\xi d\zeta dxdyds \\
& -\sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+(s,x,\xi)\alpha g_{k,2}(y,\zeta)h^k(s)d\xi d\nu^2_{y,s}(\zeta)dxdyds \\
& +\sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\alpha g_{k,1}(x,\xi)h^k(s)d\zeta d\nu^1_{x,s}(\xi)dxdyds \\
& +\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+(s,x,\xi)\partial_\zeta\alpha dm_2(y,\zeta,s)d\xi dx \\
& -\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\partial_\xi\alpha dm_1(x,\xi,s)d\zeta dy. \qquad (16)
\end{aligned}$$

Using a truncation argument of α, it is easy to see that (16) remains true if $\alpha \in C_b^\infty(\mathbb{T}^N_x \times \mathbb{R}_\xi \times \mathbb{T}^N_y \times \mathbb{R}_\zeta)$ is compactly supported in a neighbourhood of the diagonal

$$\Big\{(x,\xi,x,\xi); x\in\mathbb{T}^N, \xi\in\mathbb{R}\Big\}.$$

Taking $\alpha = \rho(x-y)\psi(\xi-\zeta)$, then we have the following remarkable identities

$$(\nabla_x + \nabla_y)\alpha = 0, \quad (\partial_\xi + \partial_\zeta)\alpha = 0. \qquad (17)$$

Referring to Proposition 13 in [12], we know that the last two terms in (16) are non-positive. Consequently, we have

$$\langle\langle f_1^+(t)\bar{f}_2^+(t),\alpha\rangle\rangle \leq \langle\langle f_{1,0}\bar{f}_{2,0},\alpha\rangle\rangle + K_1 + K_2,$$

where

$$K_1 = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi)\cdot\nabla_x + a(\zeta)\cdot\nabla_y)\alpha d\xi d\zeta dxdyds$$

and

$$\begin{aligned} K_2 \quad &= \quad -\sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^+(s,x,\xi)\alpha g_{k,2}(y,\zeta)h^k(s)d\xi d\nu^2_{y,s}(\zeta)dxdyds \\ &\quad\; +\sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s,y,\zeta)\alpha g_{k,1}(x,\xi)h^k(s)d\zeta d\nu^1_{x,s}(\xi)dxdyds \\ &:= \quad K_2^1 + K_2^2. \end{aligned}$$

By (17), we have

$$K_1 = \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) - a(\zeta))\cdot\nabla_x\alpha d\xi d\zeta dxdyds.$$

Let

$$\gamma_1(\xi,\zeta)=\int_{-\infty}^{\xi}\psi(\xi'-\zeta)d\xi',\quad \gamma_2(\zeta,\xi)=\int_{\zeta}^{\infty}\psi(\xi-\zeta')d\zeta',$$

for some $\xi,\zeta\in\mathbb{R}$. Then

$$\begin{aligned}
K_2^1 &= -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} f_1^+(s,x,\xi)\rho(x-y)\partial_\xi\gamma_1(\xi,\zeta)g_{k,2}(y,\zeta)h^k(s)d\xi d\nu^2_{y,s}(\zeta)dxdyds\\
&= -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\rho(x-y)\gamma_1(\xi,\zeta)g_{k,2}(y,\zeta)h^k(s)d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds.\\
K_2^2 &= -\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\bar{f}_2^+(s,y,\zeta)\rho(x-y)\partial_\zeta\gamma_2(\zeta,\xi)g_{k,1}(x)h^k(s)d\nu^1_{x,s}(\xi)d\zeta dxdyds\\
&= \sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\gamma_2(\zeta,\xi)\rho(x-y)g_{k,1}(x,\xi)h^k(s)d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds.
\end{aligned}$$

Note that $\gamma_1(\xi,\zeta)=\gamma_2(\zeta,\xi)=\int_{-\infty}^{\xi-\zeta}\psi(y)dy$. We deduce from (18) and (18) that

$$K_2=\sum_{k\geq 1}\int_0^t\int_{(\mathbb{T}^N)^2}\rho(x-y)\int_{\mathbb{R}^2}\gamma_1(\xi,\zeta)(g_{k,1}(x,\xi)-g_{k,2}(y,\zeta))h^k(s)d\nu^1_{x,s}\otimes d\nu^2_{y,s}(\xi,\zeta)dxdyds.$$

Hence, the equation (16) is proved for f_i^+. To obtain the result for f_i^-, we take $t_n\uparrow t$, write (16) for $f_i^+(t_n)$ and let $n\to\infty$.

Theorem 9 (Uniqueness of solutions to skeleton equations, DWZZ [9])

Let $u_0 \in L^\infty(\mathbb{T}^N)$ and assume Hypothesis H holds. Then there exists at most one kinetic solution to (13) with the initial datum u_0. Besides, any generalized solution f_h is actually a kinetic solution, i.e. if f_h is a generalized solution to (13) with initial datum $I_{u_0>\xi}$, then there exists a kinetic solution u_h to (13) with initial datum u_0 such that $f_h(x,t,\xi) = I_{u_h(x,t)>\xi}$, for a.e. (x,t,ξ).

Proof. Suppose f_1, f_2 are two kinetic solutions to the equation (15). Let ρ_γ, ψ_δ be approximations to the identity on $\mathbb{T}^N$ and $\mathbb{R}$, respectively. That is, let $\rho \in C^\infty(\mathbb{T}^N)$, $\psi \in C_c^\infty(\mathbb{R})$ be symmetric nonnegative functions such as $\int_{\mathbb{T}^N} \rho = 1$, $\int_{\mathbb{R}} \psi = 1$ and $\mathrm{supp}\psi \subset (-1,1)$. We define

$$\rho_\gamma(x) = \frac{1}{\gamma^N}\rho\Big(\frac{x}{\gamma}\Big), \quad \psi_\delta(\xi) = \frac{1}{\delta}\psi\Big(\frac{\xi}{\delta}\Big).$$

Letting $\rho := \rho_\gamma(x-y)$ and $\psi := \psi_\delta(\xi - \zeta)$ in Proposition 2.

Then

$$\begin{aligned}
&\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)dxd\xi \\
= &\int_{(\mathbb{T}^N)^2}\int_{(\mathbb{R})^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(y,t,\zeta)d\xi d\zeta dxdy + \eta_t(\gamma,\delta) \\
\leq &\int_{(\mathbb{T}^N)^2}\int_{(\mathbb{R})^2} \rho_\gamma(x-y)\psi_\delta(\xi-\zeta)f_{1,0}\bar{f}_{2,0}d\xi d\zeta dxdy + \tilde{K}_1 + \tilde{K}_2 + \eta_t(\gamma,\delta) \\
= &\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_{1,0}\bar{f}_{2,0}dxd\xi + \tilde{K}_1 + \tilde{K}_2 + \eta_t(\gamma,\delta) + \eta_0(\gamma,\delta), \qquad (18)
\end{aligned}$$

where $\tilde{K}_1$, $\tilde{K}_2$ in (18) are the corresponding K_1, K_2 in the statement of Proposition 2 with ρ, ψ replaced by ρ_γ, ψ_δ, respectively, and $\lim_{\gamma,\delta\to 0}\eta_t(\gamma,\delta)=0$, for any $t\in[0,T]$.

To complete the proof, we need to estimate $\tilde{K}_1$ and $\tilde{K}_2$.

$$|\tilde{K}_1| \leq TC_p\delta\gamma^{-1}, \quad \tilde{K}_2 \leq \sqrt{D_1}(\gamma + \delta^{\frac{1}{2}} J^{\frac{1}{2}}(\delta))(T + M).$$

Taking $\delta = \gamma^{\frac{4}{3}}$ and letting $\gamma \to 0$ gives

$$\int_{\mathbb{T}^N}\int_{\mathbb{R}} f_1^{\pm}(x,t,\xi)\bar{f}_2^{\pm}(x,t,\xi)dxd\xi \leq \int_{\mathbb{T}^N}\int_{\mathbb{R}} f_{1,0}\bar{f}_{2,0}dxd\xi.$$

Suppose that u_h^1 and u_h^2 are two kinetic solutions to (15), using the following identities

$$\int_{\mathbb{R}} I_{u_h^1>\xi}\overline{I_{u_h^2>\xi}}d\xi = (u_h^1 - u_h^2)^+, \quad \int_{\mathbb{R}} \overline{I_{u_h^1>\xi}}I_{u_h^2>\xi}d\xi = (u_h^1 - u_h^2)^-, \tag{19}$$

we deduce from (19) with $f_i = I_{u_h^i>\xi}$ that

$$\|u_h^1(t) - u_h^2(t)\|_{L^1(\mathbb{T}^N)} \leq \|u_0 - u_0\|_{L^1(\mathbb{T}^N)} = 0.$$

This gives the uniqueness.

In view of Theorem 8 and Theorem 9, we can define $\mathcal{G}^0 : C([0,T];\mathcal{U}) \to L^1([0,T];L^1(\mathbb{T}^N))$ by

$$\mathcal{G}^0(\check{h}) := \begin{cases} u_h, & \text{if } \check{h} = \int_0^{\cdot} h(s)ds, \text{ for some } h \in L^2([0,T];U), \\ 0, & \text{otherwise,} \end{cases}$$

where u_h is the solution of equation (13).

The continuity of the skeleton equations

For any family $\{h^\varepsilon; \varepsilon > 0\} \subset S_M$ and $\eta > 0$, consider the following parabolic approximation

$$\begin{cases} du^\eta_{h^\varepsilon} - \eta \Delta u^\eta_{h^\varepsilon} dt + div(A(u^\eta_{h^\varepsilon}))dt = \Phi_\eta(u^\eta_{h^\varepsilon})h^\varepsilon(t)dt, \\ u^\eta_{h^\varepsilon}(0) = u^\eta_0, \end{cases} \tag{20}$$

where u^η_0 is a smooth approximation of u_0 satisfying $\lim_{\eta \to 0} \|u^\eta_0 - u_0\|_{L^1(\mathbb{T}^N)} = 0$, Φ_η is a suitable Lipschitz approximation of Φ satisfying the linear growth condition uniformly. We define g^η_k and G^η as in the case $\eta = 0$.

Furthermore, for any $R \in \mathbb{N}$, we approximate operator A in (20) by Lipschitz continuous operator A^R using the method of truncation. Consider the following equation

$$\begin{cases} du^{\eta,R}_{h^\varepsilon} - \eta \Delta u^{\eta,R}_{h^\varepsilon} dt + div(A^R(u^{\eta,R}_{h^\varepsilon}))dt = \Phi_\eta(u^{\eta,R}_{h^\varepsilon})h^\varepsilon(t)dt, \\ u^{\eta,R}_{h^\varepsilon}(0) = u^\eta_0, \end{cases} \tag{21}$$

where A^R is Lipschitz continuous hence it has linear growth $|A^R(\xi)| \le C(R)(1 + |\xi|)$, Φ_η and u^η_0 are the same as above.

We obtain the following results:

- Using the same arguments as the proof of Theorem 5.2 in [11], for every $\eta > 0$, it gives that

$$\lim_{R\to+\infty} \sup_{\varepsilon>0} \sup_{t\in[0,T]} \|u^{\eta,R}_{h^\varepsilon}(t) - u^{\eta}_{h^\varepsilon}(t)\|^2_H = 0.$$

- $\lim_{\eta\to 0} \sup_{h\in S_M} \|u^\eta_h - u_h\|_{L^1([0,T];L^1(\mathbb{T}^N))} = 0.$
- For any $\eta, R > 0$, $\{u^{\eta,R}_\varepsilon, \varepsilon > 0\}$ is compact in $L^2([0,T];H)$.
- Fix any $\eta, R > 0$. For the solution $u^{\eta,R}_{h^\varepsilon}$ of (21), when $h^\varepsilon \to h$ weakly in $L^2([0,T];U)$,

$$\lim_{\varepsilon\to 0} \|u^{\eta,R}_{h^\varepsilon} - u^{\eta,R}_h\|_{L^1([0,T];L^1(\mathbb{T}^N))} = 0$$

  where $u^{\eta,R}_h$ is the solution of (21) with h^ε replaced by h.

Proof of (i) in Condition B

Now, we are able to prove the continuity of $\mathcal{G}^0$.

Theorem 10 (Continuity of $\mathcal{G}^0$, DWZZ [9])

Assume $h^\varepsilon \to h$ weakly in $L^2([0,T];U)$. Then u_{h^ε} converges to u_h in $L^1([0,T];L^1(\mathbb{T}^N))$, where u_{h^ε} is the kinetic solution of (15) with h replaced by h^ε.

Proof. Notice that for any $\varepsilon > 0$,

$$
\begin{aligned}
& \|u_{h^\varepsilon} - u_h\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\
\leq \; & \|u^\eta_{h^\varepsilon} - u_{h^\varepsilon}\|_{L^1([0,T];L^1(\mathbb{T}^N))} + \|u^\eta_{h^\varepsilon} - u^{\eta,R}_{h^\varepsilon}\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\
& + \|u^{\eta,R}_{h^\varepsilon} - u^{\eta,R}_{h}\|_{L^1([0,T];L^1(\mathbb{T}^N))} + \|u^{\eta,R}_{h} - u^{\eta}_{h}\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\
& + \|u^\eta_h - u_h\|_{L^1([0,T];L^1(\mathbb{T}^N))}.
\end{aligned}
$$

Proof of (ii) in Condition B

For any family $\{h^\varepsilon; 0 < \varepsilon < 1\} \subset \mathcal{A}_M$, we consider the following equation

$$\begin{cases} d\bar{u}^\varepsilon + div(A(\bar{u}^\varepsilon))dt = \Phi(\bar{u}^\varepsilon)h^\varepsilon(t)dt + \sqrt{\varepsilon}\Phi(\bar{u}^\varepsilon)dW(t), \\ \bar{u}^\varepsilon(0) = u_0. \end{cases} \tag{22}$$

By Theorem 5 and Theorem 9, we know that there exists a unique kinetic solution $\bar{u}^\varepsilon$ with initial data $u_0 \in L^\infty(\mathbb{T}^N)$ satisfying that

$$\mathbb{E}(ess \sup_{t\in[0,T]} \|\bar{u}^\varepsilon(t)\|_{L^1(\mathbb{T}^N)}) < +\infty.$$

Theorem 11 (DWZZ [9])

For every $M < \infty$, let $\{h^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$. Then

$$\left\|\mathcal{G}^\varepsilon\Big(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot h^\varepsilon(s)ds\Big) - \mathcal{G}^0\Big(\int_0^\cdot h^\varepsilon(s)ds\Big)\right\|_{L^1([0,T];L^1(\mathbb{T}^N))} \to 0 \quad \text{in probability.}$$

Proof Applying the doubling variation method with $f_1 := I_{v^\varepsilon > \xi}$ and $f_2 := I_{\bar{u}^\varepsilon > \zeta}$. Setting $\alpha(x, \xi, y, \zeta) = \rho_\gamma(x - y)\psi_\delta(\xi - \zeta)$, using integration by parts formula, we deduce that

$$
\begin{aligned}
\langle\langle f_1^+(t)\bar{f}_2^+(t), \alpha\rangle\rangle \quad = \quad & \langle\langle f_{1,0}\bar{f}_{2,0}, \alpha\rangle\rangle + \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1\bar{f}_2(a(\xi) - a(\zeta)) \cdot \nabla_x \alpha d\xi d\zeta dxdyds \\
& -\frac{\varepsilon}{2} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\zeta \alpha f_1^+(s, x, \xi) G_2^2(y, \zeta) d\bar{\nu}^{2,\varepsilon}_{y,s}(\zeta) d\xi dxdyds \\
& -\sum_{k\geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^+(s, x, \xi)\alpha g_{k,2}(y, \zeta) h^{\varepsilon,k}(s) d\xi d\bar{\nu}^{2,\varepsilon}_{y,s}(\zeta) dxdyds \\
& +\sum_{k\geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s, y, \zeta)\alpha g_{k,1}(x, \xi) h^{\varepsilon,k}(s) d\zeta d\nu^{1,\varepsilon}_{x,s}(\xi) dxdyds \\
& +\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^+(s, x, \xi)\partial_\zeta \alpha d\bar{m}_2^\varepsilon(y, \zeta, s) d\xi dx \\
& -\int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s, y, \zeta)\partial_\xi \alpha dm_1^\varepsilon(x, \xi, s) d\zeta dy \\
& -\sqrt{\varepsilon}\sum_{k\geq 1} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^+(s, x, \xi) g_{k,2}(y, \zeta)\alpha d\xi d\bar{\nu}^{2,\varepsilon}_{y,s}(\zeta) dxdyd\beta_k(s) \\
:= \quad & \langle\langle f_{1,0}\bar{f}_{2,0}, \alpha\rangle\rangle + K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7.
\end{aligned}
$$

Hard terms estimate

Verifying (i) in Condition B, we need to estimate

$$
\begin{aligned}
& K_3 + K_4 \\
\leq\ & \sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)\Big(g_{k,1}(x,\xi)h^{\varepsilon,k}(s) - g_{k,2}(y,\zeta)h^{\varepsilon,k}(s)\Big)d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds \\
\leq\ & \sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)|g_{k,1}(x,\xi) - g_{k,2}(y,\zeta)||h^{\varepsilon,k}(s)|d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds \\
\leq\ & \int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)\Big(\sum_{k\geq 1}|g_{k,1}(x,\xi) - g_{k,2}(y,\zeta)|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\geq 1}|h^{\varepsilon,k}|^2\Big)^{\frac{1}{2}}d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds \\
\leq\ & \sqrt{D_1}\int_0^t |h^\varepsilon(s)|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)|x-y|d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds \\
& +\sqrt{D_1}\int_0^t |h^\varepsilon(s)|_U \int_{(\mathbb{T}^N)^2}\rho_\gamma(x-y)\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)|\xi-\zeta|^{\frac{1}{2}}J^{\frac{1}{2}}(|\xi-\zeta|)d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^{2,\varepsilon}_{y,s}(\xi,\zeta)dxdyds \\
\leq\ & C\sqrt{D_1}(\gamma+\delta^{\frac{1}{2}}J^{\frac{1}{2}}(\delta))(t+M),
\end{aligned}
$$

where $\int_0^t |h^\varepsilon(s)|_U^2 ds \leq M$ is used.

Hard terms estimate

Verifying (a) in Condition A, we need to estimate

$$
\begin{aligned}
& K_3 + K_4 \\
\leq \; & \sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)\Big(g_{k,1}(x)h^{\varepsilon,k}(s) - g_{k,2}(y)h^k(s)\Big)d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^2_{y,s}(\xi,\zeta)dxdyds \\
\leq \; & \sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)|g_{k,1}(x) - g_{k,2}(y)||h^{\varepsilon,k}(s)|d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^2_{y,s}(\xi,\zeta)dxdyds \\
& + \sum_{k\geq 1}\int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)|g_{k,2}(y)||h^{\varepsilon,k}(s) - h^k(s)|d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^2_{y,s}(\xi,\zeta)dxdyds \\
\leq \; & \int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)\Big(\sum_{k\geq 1}|g_{k,1}(x) - g_{k,2}(y)|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\geq 1}|h^{\varepsilon,k}(s)|^2\Big)^{\frac{1}{2}}d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^2_{y,s}(\xi,\zeta)dxdyds \\
& + \int_0^t \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2} \gamma_1(\xi,\zeta)\rho_\gamma(x-y)\Big(\sum_{k\geq 1}|g_{k,2}(y)|^2\Big)^{\frac{1}{2}}\Big(\sum_{k\geq 1}|h^{\varepsilon,k}(s) - h^k(s)|^2\Big)^{\frac{1}{2}}d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^2_{y,s}(\xi,\zeta)dxdyds \\
\leq \; & C\sqrt{D_1}(\gamma + \delta^{\frac{1}{2}}J^{\frac{1}{2}}(\delta))(t+M) + \sqrt{D_0}\int_0^t |h^\varepsilon - h|_U \int_{(\mathbb{T}^N)^2}\int_{\mathbb{R}^2}\gamma_1(\xi,\zeta)\rho_\gamma(x-y)d\nu^{1,\varepsilon}_{x,s}\otimes d\bar{\nu}^2_{y,s}(\xi,\zeta)dxdyds \\
\leq \; & C\sqrt{D_1}(\gamma + \delta^{\frac{1}{2}}J^{\frac{1}{2}}(\delta))(t+M) + Q.
\end{aligned}
$$

Compared with Condition B, we need to show that the additional term Q converges to 0, when $h^\varepsilon(s)$ weakly converges to $h(s)$. Unfortunately, we failed! Because, we don't have some nice moment estimates, such as $L^2([0,T];V)$, V is a suitable Sobolev space.

$$
\begin{aligned}
|K_1| &\leq tC_p\delta\gamma^{-1}, \quad K_5+K_6\leq 0,\ K_2\leq \frac{\varepsilon}{2}D_0t\delta^{-1}+\frac{\varepsilon}{2}D_0t\delta. \\
K_3+K_4 &\leq C\sqrt{D_1}(\gamma+\delta^{\frac{1}{2}}J^{\frac{1}{2}}(\delta))(t+M),\ \mathbb{E}\sup_{t\in[0,T]}|K_7|(t)\leq C\sqrt{\varepsilon}\sqrt{D_0T}\gamma^{-N}.
\end{aligned}
$$

Letting $\delta=\gamma^{\frac{4}{3}}, \quad \gamma=\varepsilon^{\frac{1}{2(1+N)}}$, then, $\mathbb{E}\sup_{t\in[0,T]}|K_7|(t)\to 0 \quad \varepsilon\to 0$, which implies that $\sup_{t\in[0,T]}|K_7|(t)\to 0$ in probability as $\varepsilon\to 0$ by Chebyshev inequality. Moreover,

$$
\sup_{t\in[0,T]} r(\varepsilon,\gamma,\delta,t)\to 0, \quad a.s. \quad \varepsilon\to 0.
$$

Notice that $f_1:=I_{v^\varepsilon>\xi}$ and $f_2:=I_{\bar{u}^\varepsilon>\zeta}$ with initial data $f_{1,0}=I_{u_0>\xi}$ and $\bar{f}_{2,0}=I_{u_0>\zeta}$, respectively. With the help of identity (19), we deduce that

$$
\|\bar{u}^\varepsilon(t)-v^\varepsilon(t)\|_{L^1(\mathbb{T}^N)}\leq |K_7|(t)+r(\varepsilon,t).
$$

Hence, it follows that

$$
\begin{aligned}
&\|\bar{u}^\varepsilon-v^\varepsilon\|_{L^1([0,T];L^1(\mathbb{T}^N))} \\
\leq\ & T\cdot ess\sup_{t\in[0,T]}\|\bar{u}^\varepsilon(t)-v^\varepsilon(t)\|_{L^1(\mathbb{T}^N)} \\
\leq\ & T\cdot \sup_{t\in[0,T]}|K_7|(t)+T\cdot\sup_{t\in[0,T]} r(\varepsilon,t)\to 0
\end{aligned}
$$

in probability as $\varepsilon\to 0$. We thus complete the proof.

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Thank you!